The Phase Transition in Heuristic Search

J. Christopher Beck
Department of Mechanical & Industrial Engineering
University of Toronto
Canada
jcb@mie.utoronto.ca
The lack of interest, the distain for history is what makes computing not-quite-a-field.

- Alan Kay, Dr. Dobbs, July 10, 2012

Nothing is as good as it used to be, and it never was. The “golden age of sports,” the golden age of anything, is the age of everyone’s childhood.

- Ken Dryden, “The Game”

Corollary: The best papers are the ones we read during grad school.
Outline

• The Phase Transition
  – aka Flashback to the 1990s

• The Phase Transition in Heuristic Search
  – An abstract model and benchmark problems

• The Effect of Operator Cost Ratio

• Next Steps
  – Heavy-Tails and Local Minima?
Where the Hard Problems Are

• While NP problems are worst-case exponential to solve, often typical instances are practically solvable
• Q: What is the distribution of the empirically hard instances?
Graph Coloring

Graph Coloring

(b) 3-color difficulty

Conjectures

- All NP-complete problems have an “order parameter” (TSP, CSP, SAT, HC, ...)
- A critical value of the order parameter separates regions of under-constrained and over-constrained problem instances
- The hard problem instances are found around this critical value

Random 3-SAT

% Solubility and Normalized difficulty

Why Do We Care?

• A lot of recent interest in understanding the difficulty of heuristic search problems – i.e., “A*-style” state-based search

• The phase transition has not (yet) been shown for heuristic search problems

Does the phase transition phenomenon play a role in problem difficulty for heuristic search?
Some more background …
State-Space Search (aka “Heuristic Search”)

Greedy Best-First Search (GBFS):
choose node with minimum h

Possible transitions

Path from node to goal (estimate): h = 5
PT in Planning

• Randomly generate planning problems
  – operators, preconditions, effects, ...

• Bylander [AIJ 1996]
  – Bounds based on goals and atoms to operators ratio

• Rintanen [KR 2004]
  – Gradual transition between soluble and insoluble based on operator/variable ratio
  – Hampered by lack of insolubility test
Quantified SAT (2-QSAT)

• Gent & Walsh [AAAI 1999]
  – apply theory of “constrainedness” from NP to PSPACE
  – PT and easy-hard-easy observed for 2-QSAT once trivially insoluble instances removed
  – More convincing evidence of abrupt PT than in the planning work
Problem Difficulty for GBFS

• Operator cost ratio
  – higher ratio $\approx$ more effort
    • (but see Fan et al. ICAPS2017)

• Unininformative Heuristic Regions (UHRs)
  – plateaux and local minima $\approx$ more effort

• Correlation between heuristic and distance
  – lower correlation $\approx$ more effort

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Abstract Model

Model 1. Let \( n \in \mathbb{Z}^+ \) be the number of states in the problem space \( S = \{s_1, s_2, \ldots, s_n\} \) and \( p \in [0, 1] \) be the connectivity density of the problem space. The class \( Q_{n,p} \) consists of all problem instances \( \langle T, S_i, S_g \rangle \) such that:

1. \( T \) is a random transition graph drawn from \( D_{n,p} \), the probability space of all random digraphs (Karp 1990).
2. \( S_i \in S \) is a randomly chosen initial state such that \( \exists k \neq i : (S_i, S_k) \in T \).
3. \( S_g \in S \) is a randomly chosen goal state such that \( S_g \neq S_i \) and \( \exists k \neq g : (S_k, S_g) \in T \).

Control Parameter

\[
\gamma := \frac{\text{Expected number of edges in the transition graph}}{\text{Number of states}}
\]
Solubility

Solubility: 0.1% to 99.9%

Is this surprising?
# Nodes Expanded

- **50%**
- **100%**
- **Crossover**
- **Bounds**
- **Mushy Region**
Effect of the Heuristic

A new question:
What is the impact of systematically stronger heuristics?

True cost to goal

\[ h_i(x) = \begin{cases} c(x), & \text{if } c(x) < i \\ i, & \text{otherwise} \end{cases} \]
Effect of the Heuristic

Soluble instances only
Abstract Model

- Solubility phase transition
- Easy-hard-easy pattern associated with PT
- New results on the impact of heuristics across PT

Standard PT work (CP, SAT) uses an abstract model on random problems analogous to ours.

What about benchmark problems?
Benchmarks

• Given an existing benchmark problem, we can generate relaxed/restricted instances by adding/removing transitions
Benchmarks

Definition 3. (Observed connectivity density) Let \( G(V, E) \) be an arbitrary transition graph. We define the observed connectivity density of this graph \( \mathcal{P}(G) = \frac{|E|}{|V| \cdot (|V| - 1)} \).

Model 2. Given an existing problem’s transition graph \( G(V, E) \) and the required connectivity density \( p \), the class \( R_{G,p} \) consists of all problem instances \( \langle T, S_i, S_g \rangle \) such that:

1. \( T \), the transition graph, is a restricted instance of \( G \) if \( p < \mathcal{P}(G) \), or a relaxed instance otherwise. \( \mathcal{P}(T) = p \).
2. \( S_i \in S \), a randomly chosen initial state, \( \exists k : (S_i, S_k) \in T \)
3. \( S_g \in S \), a randomly chosen goal state such that \( S_g \neq S_i \) and \( \exists k : (S_k, S_g) \in T \)
The Pancake Problem

Action $F_k$: flip top $k$

initial state

Solution: $F_5, F_6, F_3, F_4, F_5$

goal state

The Pancake Problem

Figure 2: 8-Pancake Problem: Solubility and search effort (50% and 100% percentile) plotted against $\gamma$ (log-log scale).
The Grid Navigation Problem
The Grid Navigation Problem

Figure 3: $150 \times 150$ Grid Navigation Problem: Solubility and search effort plotted against $\gamma$ (log-log scale).
Similar Results

• TopSpin
• Towers of Hanoi
• Interesting differences with 8 Sliding Tile Puzzle due to disconnected search space
Effect of Heuristic (8-Pancake)
So ...

• Phase transition and easy-hard-easy patterns exist in GBFS for both abstract model and benchmark problems

• Heuristics of systematically increasing strengths show radically different performance across the phase transition
  – Lowest improvement on hardest problems

What about existing ideas about problem difficulty in heuristic search?
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Operator Cost Ratio

• [Wilt & Ruml 2011]
  – Instances are far more difficult with non-unit costs despite the same connection structure

• [Cushing et al. 2011]
  – Cost variance fundamentally misleads heuristic search

• [Fan et al. 2017]
  – No Free Lunch Theorem for Dijkstra’s Alg.
    • Negative effects are balanced by positive effects in other cost functions
Operator Cost Ratio and the PT

What is the impact of the operator cost ratio on problem difficulty across relaxed/restricted benchmark problems?

Grid Navigation

\[ C_m(s, a) = \begin{cases} 
1^m, & \text{if } a = \text{up} \\
2^m, & \text{if } a = \text{down} \\
3^m, & \text{if } a = \text{left} \\
4^m, & \text{if } a = \text{right}
\end{cases} \]
Grid Navigation

![Graph showing grid navigation data]

Figure 3: 500 × 500 Grid Navigation: Median effort ratio of soluble instance vs. probability of an unblocked cell.
Pancake Problem

- Cost = $z^m$
  - $z$: size of the bottom pancake in flipped sub-pile
- For the 8-Pancake problem the operator cost ratio is $8^m$

Figure 5: 8-Pancake Problem: Median effort ratio of soluble instance vs. $\gamma$. 
[Wilt & Ruml 2014] for TopSpin, sometimes higher operator cost ratio is better.

Figure 7: 10-disk Top: Median effort ratio of soluble instances vs. $\gamma$. 
Operator Cost Ratio and the PT

• Impact of higher operator cost ratio follows a low-high-low pattern, peaking in the PT
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The Pancake Problem

Figure 2: 8-Pancake Problem: Solubility and search effort (50% and 100% percentile) plotted against $\gamma$ (log-log scale).
Pancake Problem (Median)

Figure 5: 8-Pancake Problem: Median effort ratio of soluble instance vs. $\gamma$. 
Pancake Problem

“Exceptionally hard problems (ehps)” [Gent & Walsh 1994]

Figure 13: 8-Pancake: 99.9%-percentile effort vs. $\gamma$. 
Exceptionally Hard Problems

• Very hard problems in underconstrained regions of the PT
• Not inherently hard problems
  – Combination of problem structure and algorithm details
• Heavy-tailed distributions
  – Performance of randomized heuristic follows a heavy-tailed distribution

Heavy-Tailed Runtime Distributions

Figure 1: Log-log plot of the tail of 12 team round-robin scheduling.

Failed Sub-trees and Local Minima

- Failed sub-tree (CSP)
  - A sub-tree with no solutions
  - If entered (e.g. by depth-first search) needs to be exhaustively searched

- Local Minima (heuristic search)
  - [Wilt & Ruml 2014]
  - A region that does not contain the goal but that the search will have to exhaust if it enters
  - Connected with difficulty due to higher operator cost ratio

Heavy-tails occurs when depths of failed sub-trees are exponentially distributed [Gomes et al. 2005]
Problem Difficulty for GBFS

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So What Have We Done?

• Showed that the phase transition phenomenon from combinatorial search can be observed in heuristic search

• Showed an (empirical) relation between PT and problem hardness
  – Both unit-cost problems and when varying operator cost ratio

• Showed the existence of *ehps* for GBFS
Conjectures

• The size and extend of local minima is effected by the phase transition
• The analysis of problem difficulty based on heavy-tailed distributions (in CSPs) can be imported into heuristic search
Science requires a society because even people who are trying to be good thinkers love their own thoughts and theories – much of the debugging has to be done by others.